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Excitations in a Dipolar Bose–Einstein Condensate

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Abstract We study excitations in a dipolar Bose–Einstein condensate with Green's function. In Bogoliubov approximation, we obtain the dispersion relation. The excitation energy is dependent on the angle between the momentum and the magnetic moment. In the long-wave limit, the dispersion relation reduces to an anisotropic phonon-like dispersion relation.

Keywords Dipolar Bose–Einstein condensate · Excitation · Green function

1 Introduction

In degenerate atomic quantum gases, the underlying physics are strongly determined by inter-atomic interactions. Short-range and isotropic interactions have been extensively considered in the past decade [1–3]. Until recently, the experimental realization of Bose–Einstein condensation in strongly magnetic ^{52}Cr atoms [4] opens a fascinating new research area, namely that of dipolar gases, for which the dipole–dipole interaction plays a significant role. This experimental breakthrough makes it possible to study the properties of long-ranged and anisotropic atomic gases, so that it attracts much attention of experimental and theoretical physicists.

Ground-state solutions in a dilute gas interacting via contact and magnetic dipole–dipole forces have been investigated [5]. Góral and Santos have studied the ground-state properties of Bose–Einstein condensates of trapped dipolar gases and their low-lying collective excitations by analyzing numerically the response of a BEC after applying an external perturbation [6]. Lushnikov have studied the dynamics of a dipolar Bose–Einstein condensate in a harmonic trap [7]. Yi and You have discussed ground-state properties of trapped atomic condensates with electric-field induced dipole–dipole interactions [8], properties of trapped

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atomic condensates with anisotropic dipole interactions [9], and the low-energy shape oscillations of a magnetically trapped atomic condensate including the spin dipole interaction [10]. Santos et al. have investigated Roton–Maxon spectrum and stability of trapped dipolar Bose–Einstein condensates [11]. O'Dell et al. have presented exact results in the Thomas–Fermi regime for the statics and dynamics of a harmonically trapped dipolar Bose–Einstein condensate [12]. Cooper et al. have studied the effects of dipolar interactions on the groundstate of a rotating atomic Bose gas [13]. Xie et al. have presented the different types of solitary excitations of a dipolar Bose–Einstein condensate in an optical lattice [14].

In this paper, we consider the system of bosonic atoms with dipole–dipole interactions at zero temperature. We assume that the interaction between atoms is weak enough to treat the system with perturbative method. In zero temperature, the excited atoms are very dilute, so we can neglect the interactions between excited atoms, i.e. we take Bogoliubov approximation in our system. We will obtain the dispersion relation in this approximation and analyze its properties.

2 The Model

For the system of spinless bosonic atoms, the grand cannonical Hamiltonian in the second quantized formalism is given by

$$\begin{aligned}\hat{H} = \int d\boldsymbol{r}\biggl[&\frac{\hbar^2}{2m}\nabla\Psi^\dagger(\boldsymbol{r})\cdot\nabla\Psi(\boldsymbol{r}) - \mu\Psi^\dagger(\boldsymbol{r})\Psi(\boldsymbol{r})\\ &+\frac{1}{2}\int d\boldsymbol{r}'\Psi^\dagger(\boldsymbol{r})\Psi^\dagger(\boldsymbol{r}')V(\boldsymbol{r}-\boldsymbol{r}')\Psi(\boldsymbol{r}')\Psi(\boldsymbol{r})\biggr],\end{aligned} \tag{1}$$

where $\Psi(\boldsymbol{r})$ and $\Psi^\dagger(\boldsymbol{r})$ are bosonic annihilation and creation operators respectively and satisfy the commutation relation $[\Psi(\boldsymbol{r}), \Psi^\dagger(\boldsymbol{r}')] = \delta(\boldsymbol{r}-\boldsymbol{r}')$, m is the mass of atom and μ is the chemical potential. $V(\boldsymbol{r})$ is the interaction potential between atoms, for dipolar atoms, which can be divided into two parts as

$$V(\boldsymbol{r}) = V_0(\boldsymbol{r}) + V_d(\boldsymbol{r}). \tag{2}$$

Here $V_0(\boldsymbol{r})$ is the bare potential described by the pseudopotential as

$$V_0(\boldsymbol{r}) = g\delta(\boldsymbol{r}), \tag{3}$$

with $g = 4\pi\hbar^2 a_s/m$, where a_s is the scattering length of atom and m is the mass of atom. $V_d(\boldsymbol{r})$ is the dipole interaction between atoms, which is described by

$$V_d(\boldsymbol{r}) = \frac{\mu_0\mu_m^2}{4\pi r^3}\left(1 - \frac{3(\mathbf{e}_\mu\cdot\boldsymbol{r})^2}{r^2}\right), \tag{4}$$

where μ_0 is magnetic permeability of the vacuum and μ_m is the magnetic moment of atom, $\mathbf{e}_\mu$ is the unit vector directed along the magnetic moment.

$$V_d(\boldsymbol{r}) = \eta g\frac{1-3\cos^2\theta}{r^3}, \tag{5}$$

where $\eta = m\mu_0\mu_m^2/16\pi^2\hbar^2 a_s$ represents the ratio of the dipole–dipole interaction and the hard core interaction.

We take a Fourier transformation to the field operators as

$$\Psi(\boldsymbol{r}) = \frac{1}{\sqrt{v}} \sum_k \hat{a}_k e^{ik\cdot r}, \tag{6}$$

$$\Psi(\boldsymbol{r})^\dagger = \frac{1}{\sqrt{v}} \sum_k \hat{a}_k^\dagger e^{-ik\cdot r}, \tag{7}$$

where v is the volume of the system, a_k and $a_k^\dagger$ are annihilation and creation operators, which destroy and create one particle state of plane waves with momentum $\boldsymbol{k}$, respectively. These operators are bosonic in our case, so they satisfy the commutation relations:

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}, \tag{8}$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \tag{9}$$

Similarly, we take a Fourier transformation to the interaction potential as

$$U(\boldsymbol{q}) = \int d\boldsymbol{r} e^{-iq\cdot r} V(\boldsymbol{r}),$$

which can be divided into the bare potential

$$U_0 = \int d\boldsymbol{r} e^{-iq\cdot r} g\delta(\boldsymbol{r}) = g,$$

and the dipole–dipole potential

$$U_d(\boldsymbol{q}) = \int d\boldsymbol{r} e^{-iq\cdot r} V_d(\boldsymbol{r}).$$

The Fourier transform of the dipole–dipole potential can be written as [5, 6]

$$\mathcal{F}[V_d](\boldsymbol{q}) = 4\pi\eta g(1 - 3\cos^2\alpha)\left[\frac{\cos(bq)}{(bq)^2} - \frac{\sin(bq)}{(bq)^3}\right], \tag{10}$$

where $\mathcal{F}$ indicates the Fourier transform, α is the angle between the momentum $\boldsymbol{q}$ and the dipole direction, and b is a cutoff distance corresponding to the atomic radius. In fact, b is much smaller than any significant length scale of the system, so one take the limit as [5, 6]

$$U_d(\boldsymbol{q}) = \lim_{b\to 0} \mathcal{F}(V_d(\boldsymbol{r})) = \frac{4\pi}{3}\eta g(3\cos^2\alpha - 1). \tag{11}$$

In the above formalism, the Hamiltonian (1) can be rewritten as,

$$\hat{H} = \sum_k (\epsilon_k - \mu) a_k^\dagger a_k + \frac{1}{2v} \sum_{q,k,k'} U(\boldsymbol{q}) \hat{a}_{k+q}^\dagger \hat{a}_{k'-q}^\dagger \hat{a}_{k'} \hat{a}_k, \tag{12}$$

where $\epsilon_k = \hbar^2 k^2/2m$. We introduce a new set of bosonic annihilation and creation operators with a canonical transformation:

$$\hat{b}_k = \hat{a}_k - \delta_{k,0}\sqrt{N_0}, \tag{13}$$

$$\hat{b}_k^\dagger = \hat{a}_k^\dagger - \delta_{k,0}\sqrt{N_0}. \tag{14}$$

The relation between the chemical potential and the condensate density can be derived from the requirement that $\langle b_k \rangle = \langle b_k^\dagger \rangle = 0$. Substituting the above equations into (12), one obtains

$$H = \sum_{k\neq 0}(\epsilon_k - \mu)b_k^\dagger b_k + \frac{1}{2}n_0^2 vU(0) - \mu N_0 + \frac{1}{2}n_0 \sum_k U(\boldsymbol{k})b_k b_{-k} + \frac{1}{2}n_0 \sum_k U(\boldsymbol{k})b_k^\dagger b_{-k}^\dagger$$
$$+ n_0 \sum_k U(\boldsymbol{k})b_k^\dagger b_k + n_0 \sum_k U(0)b_k^\dagger b_k + \frac{n_0^{1/2}}{v^{1/2}} \sum_{k,q} U(\boldsymbol{q})b_{k+q}^\dagger b_k b_q$$
$$+ \frac{n_0^{1/2}}{v^{1/2}} \sum_{k,q} U(\boldsymbol{q})b_k^\dagger b_q^\dagger b_{k+q} + \frac{1}{2v} \sum_{k,k',q} U(\boldsymbol{q})b_{k+q}^\dagger b_{k'-q}^\dagger b_{k'} b_k, \tag{15}$$

where $n_0 = N_0/v$ is the number density of condensed bosonic atoms.

3 Green's Function

In the τ-dependent Heisenberg picture, the field operator b_k and $b_k^\dagger$ can be rewritten as

$$b_k(\tau) = e^{H\tau} b_k e^{-H\tau}, \tag{16}$$
$$b_k^\dagger(\tau) = e^{H\tau} b_k^\dagger e^{-H\tau}, \tag{17}$$

where the variable τ can be viewed as an imaginary time $\tau = it/\hbar$ comparing with the standard real-time representation. In the case of a uniform system, the single-particle thermal Green's function is defined as [15, 16]

$$\mathcal{G}(\boldsymbol{k}, \tau) = -\langle T_\tau b_k(\tau) b_k^\dagger(0)\rangle, \tag{18}$$

where T_τ is a τ ordering operator, and $\langle \cdots \rangle = \mathrm{Tr}[e^{\beta(\Omega - H)} \cdots]$ is the grand canonical ensemble average with $\beta = 1/k_b T$ and Ω being the thermodynamic potential. Correspondingly, we can define the anomalous Green's function as

$$\mathcal{G}_{12}(\boldsymbol{k}, \tau) = -\langle T_\tau b_k(\tau) b_k(0)\rangle, \tag{19}$$
$$\mathcal{G}_{21}(\boldsymbol{k}, \tau) = -\langle T_\tau b_k^\dagger(\tau) b_k^\dagger(0)\rangle, \tag{20}$$

which represent the disappearance and appearance of two non-condensate particles, respectively.

For simplicity, we introduce a matrix operator

$$\hat{B}_k = \begin{bmatrix} \hat{b}_k \\ \hat{b}_k^\dagger \end{bmatrix} \tag{21}$$

and, correspondingly, define a 2×2 matrix Green's Function

$$\mathbf{G}(\boldsymbol{k}, \tau) = \langle T_\tau [\hat{B}_k(\tau) \hat{B}_k^\dagger(0)]\rangle. \tag{22}$$

After taking Fourier transformation, we obtain

$$\mathbf{G}(\boldsymbol{k}, i\omega_n) = \frac{1}{\beta\hbar} \int_0^{\beta\hbar} d\tau e^{i\omega_n \tau} \mathbf{G}(\boldsymbol{k}, \tau). \tag{23}$$

The Dyson–Beliaev equation can be written as follow

$$\mathbf{G}(\boldsymbol{k}, i\omega_n) = \mathbf{G}^{(0)}(\boldsymbol{k}, i\omega_n) + \mathbf{G}^{(0)}(\boldsymbol{k}, i\omega_n)\Sigma(\boldsymbol{k}, i\omega_n)\mathbf{G}(\boldsymbol{k}, i\omega_n). \tag{24}$$

Here, the matrix Green's function is

$$\mathbf{G}(\boldsymbol{k}, i\omega) = \begin{bmatrix} \mathcal{G}_{11}(\boldsymbol{k}, i\omega) & \mathcal{G}_{12}(\boldsymbol{k}, i\omega) \\ \mathcal{G}_{21}(\boldsymbol{k}, i\omega) & \mathcal{G}_{22}(\boldsymbol{k}, i\omega) \end{bmatrix}, \tag{25}$$

where the matrix elements are not independent, with the following relations:

$$\mathcal{G}_{11}(\boldsymbol{k}, i\omega) = \mathcal{G}(\boldsymbol{k}, i\omega), \tag{26}$$

$$\mathcal{G}_{22}(\boldsymbol{k}, i\omega) = \mathcal{G}_{11}(-\boldsymbol{k}, -i\omega), \tag{27}$$

$$\mathcal{G}_{12}(\boldsymbol{k}, i\omega) = \mathcal{G}_{21}(-\boldsymbol{k}, -i\omega). \tag{28}$$

The unperturbed matrix Green's function is

$$\mathbf{G}^{(0)}(\boldsymbol{k}, i\omega) = \begin{bmatrix} \mathcal{G}^{(0)}(\boldsymbol{k}, i\omega) & 0 \\ 0 & \mathcal{G}^{(0)}(-\boldsymbol{k}, -i\omega) \end{bmatrix}, \tag{29}$$

where $\mathcal{G}^{(0)}(\boldsymbol{k}, i\omega)$ is given by

$$\mathcal{G}^{(0)}(\boldsymbol{k}, i\omega_n) = \frac{1}{i\omega_n - \epsilon_k + \mu}. \tag{30}$$

The proper self-energy matrix is

$$\Sigma(\boldsymbol{k}, i\omega) = \begin{bmatrix} \Sigma_{11}(\boldsymbol{k}, i\omega) & \Sigma_{12}(\boldsymbol{k}, i\omega) \\ \Sigma_{21}(\boldsymbol{k}, i\omega) & \Sigma_{22}(\boldsymbol{k}, i\omega) \end{bmatrix}, \tag{31}$$

where the matrix elements are not independent, with the following relations,

$$\Sigma_{22}(\boldsymbol{k}, i\omega) = \Sigma_{11}(-\boldsymbol{k}, -i\omega), \tag{32}$$

$$\Sigma_{12}(\boldsymbol{k}, i\omega) = \Sigma_{21}(-\boldsymbol{k}, -i\omega). \tag{33}$$

Solving (24), one obtains the following solution [15, 16]:

$$\mathcal{G}_{11}(\boldsymbol{k}, i\omega) = \frac{i\omega_n + \epsilon_k - \mu + \Sigma_{11}(-\boldsymbol{k}, -i\omega)}{[i\omega_n + \epsilon_k - \mu + \Sigma_{11}(-\boldsymbol{k}, -i\omega)][i\omega_n - \epsilon_k + \mu - \Sigma_{11}(\boldsymbol{k}, i\omega)] + [\Sigma_{12}(\boldsymbol{k}, i\omega)]^2}, \tag{34}$$

$$\mathcal{G}_{12}(\boldsymbol{k}, i\omega) = \frac{-\Sigma_{12}(\boldsymbol{k}, i\omega)}{[i\omega_n + \epsilon_k - \mu + \Sigma_{11}(-\boldsymbol{k}, -i\omega)][i\omega_n - \epsilon_k + \mu - \Sigma_{11}(\boldsymbol{k}, i\omega)] + [\Sigma_{12}(\boldsymbol{k}, i\omega)]^2}. \tag{35}$$

Here, $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ are the normal and anomalous Green's functions in terms of the exact proper self-energies.

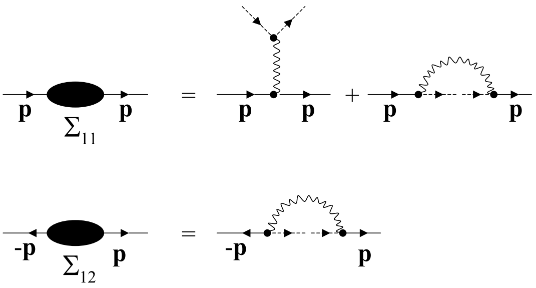

Fig. 1 Proper self-energy diagrams in the Bogoliubov approximation. A *solid line* with *arrow* denotes a free non-condensed boson, a *dashed line* denotes a condensed boson and a *wiggly line* denotes the interactions including the hard core interaction and the dipole–dipole interaction

4 Excitation Spectrum

In this section, we shall calculate the Green's functions in the Bogoliubov approximation. This approximation is valid at very low temperature, where all the terms coming from non-condensate density can be neglected. The Feynman diagrams of proper self-energy in the Bogoliubov approximation are shown in Fig. 1. The proper self-energies in the Bogoliubov approximation are

$$\Sigma_{11}(\boldsymbol{k}, i\omega) = n_0[U(0) + U(\boldsymbol{k})], \tag{36}$$

$$\Sigma_{12}(\boldsymbol{k}, i\omega) = n_0 U(\boldsymbol{k}), \tag{37}$$

where $n_0 = N_0/v$ is the condensate density. With the Hugenholtz–Pines theorem [17], the chemical potential only containing the lowest-order contribution is given by

$$\mu = n_0 U(0). \tag{38}$$

Substituting (36–38) into (34) and (35), one obtains

$$\mathcal{G}_{11} = \frac{i\omega_n + \epsilon_k + n_0 U(\boldsymbol{k})}{(i\omega)^2 - \epsilon_k^2 - 2n_0 U(\boldsymbol{k})\epsilon_k}, \tag{39}$$

$$\mathcal{G}_{12} = \frac{-n_0 U(\boldsymbol{k})}{(i\omega)^2 - \epsilon_k^2 - 2n_0 U(\boldsymbol{k})\epsilon_k}. \tag{40}$$

Setting the denominators of the above Greens function as zero and replacing $i\omega$ with E_k, one obtains the following dispersion relation:

$$E_k = \sqrt{\epsilon_k^2 + 2n_0 U(\boldsymbol{k})\epsilon_k}, \tag{41}$$

where $U(\boldsymbol{k}) = U_0 + U_d(\boldsymbol{k})$. From (11), we know that $U_d(\boldsymbol{k})$ is only dependent on the angle of the momentum $\boldsymbol{k}$ and the magnetic moment of atom, so the dispersion relation can rewritten as,

$$E_k = \sqrt{\epsilon_k^2 + 2n_0[U_0 + U_d(\alpha)]\epsilon_k}, \tag{42}$$

with the s-scattering length a_s as the characterizing length, we obtain the dimensionless dispersion relation as follows,

$$\xi_k = \sqrt{k_0^4 + 2[u_0 + u_d(3\cos^2\alpha - 1)]k_0^2}, \tag{43}$$

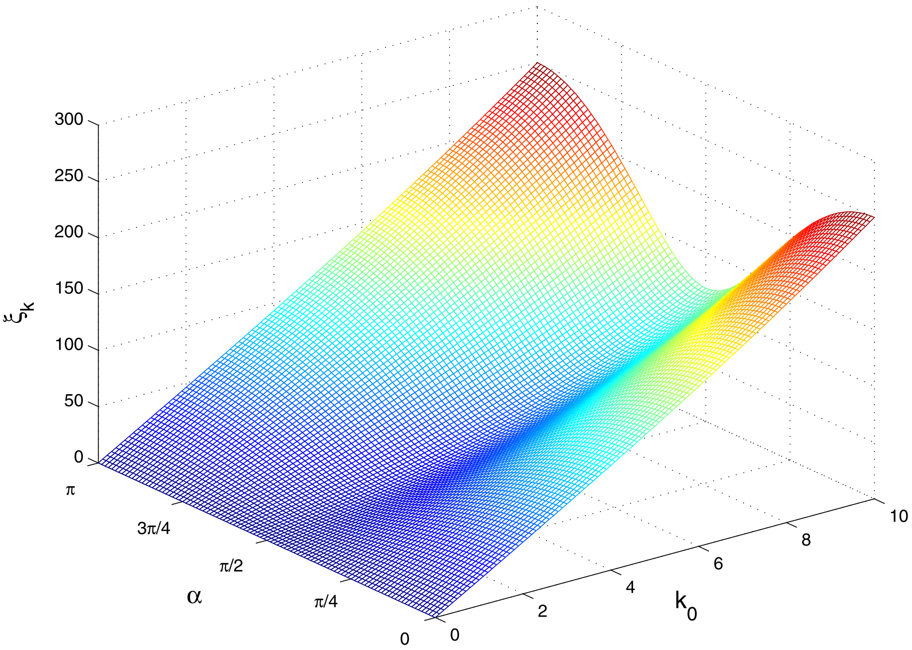

Fig. 2 Diagram for the dimensionless dispersion relation with $u_0 = 1$ and $u_d = 0.8$

where $\xi_k = 2ma_s^2 E_k/\hbar^2$ is the dimensionless energy, $k_0 = ka_s$ is the dimensionless wave vector, and $u_0 = 8\pi a_s^3 n_0$, $u_d = 32\pi^2 a_s^2 n_0 \eta/3$ are the dimensionless interaction parameters. Figure 2 shows the change of dimensionless energy ξ_k with the dimensionless wave vector k_0 and the angle α between the wave vector and the magnetic moment.

In the long-wavelength limit ($|\boldsymbol{k}| \to 0$), E_k reduces to the anisotropic phonon-like dispersion relation

$$E_k|_{|k|\to 0} \sim \hbar c(\alpha) k, \tag{44}$$

with the anisotropic characteristic velocity

$$c(\alpha) = \sqrt{n_0[U_0 + U_d(\alpha)]/m}. \tag{45}$$

We note that the phonon is anisotropic and dependent on the direction of momentum, which is different from the atomic system with isotropic interaction.

5 Conclusion

In this paper, we investigated the excitation spectrum of a dipolar Bose–Einstein condensate with by using the Green's function method. In the zero temperature, the most atoms are condensed and the excited atoms are very dilute, so we did not consider the interaction between the excited atoms, i.e. Bogoliubov approximation. In this approximation, we obtained the Green's function by solving the Dyson–Beliaev equations. By setting the denominators of the Green's function as zero, we obtained the dispersion relation and noted that the dispersion relation is anisotropic. In the long-wavelength limit, the dispersion relation reduces to the anisotropic phonon-like dispersion relation.

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